Central extensions of groups of symplectomorphisms

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We construct canonically defined central extensions of groups of symplectomorphisms. We show that this central extension is nontrivial in the case of a torus of dimension ≥ 6 and in the case of a two-dimensional surface of genus ≥ 3.

1 Formulation of results

Central extensions of the groups of symplectomorphisms discussed in this paper appeared as a byproduct in [25]. Here we prove several nontrivality and triviality theorems concerning this cocycle.

1.1. Preliminaries. Cocycle on the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$**.** We define the real symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ as the group of real matrices

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

preserving the standard skew-symmetric bilinear form $K := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, i.e.

$$g^t K g = K \tag{1.2}$$

The complex symplectic group $\mathrm{Sp}(2n,\mathbb{C})$ is the group of complex matrices satisfying the same condition (1.2).

Consider the block $(n+n)\times(n+n)$ matrix $J \in \mathrm{Sp}(2n,\mathbb{C})$ given by

$$J = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{1.3}$$

For $g \in \mathrm{Sp}(2n,\mathbb{R})$, we consider the matrix $J^{-1}gJ \in \mathrm{Sp}(2n,\mathbb{C})$, this matrix has the structure

$$J^{-1}gJ = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}$$

where bar means the element-wise complex conjugation. We denote

$$\Phi = \Phi(g); \qquad \Psi = \Psi(g)$$

We define the *Berezin cocycle*

$$c : \mathrm{Sp}(2n,\mathbb{R}) \times \mathrm{Sp}(2n,\mathbb{R}) \to \mathbb{R}$$

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by[2]

$$c(g_1,g_2) = \mathrm{Im}\ \mathrm{tr}\ \ln\Bigl[\Phi(g_1)^{-1}\Phi(g_1g_2)(\Phi(g_2)^{-1}\Bigr] \tag{1.4}$$

Below (Theorem 2.1) we show that the matrix in the brackets has the form $1+Z$, where $\|Z\|<1$. Then $\ln(1+Z) := Z - Z^2/2 + Z^3/3 - \dots$ and hence our expression is well defined.

The cocycle c defines a central extension of $\mathrm{Sp}(2n,\mathbb{R})$. In other words, the set $\mathrm{Sp}(2n,\mathbb{R})\times\mathbb{R}$ with the multiplication

$$(g,x)\cdot(h,y) = (gh, x+y+c(g,h))$$

is a group.

1.2. Groups of symplectomorphisms. Notation. Consider a $2n$-dimensional symplectic manifold M. Define the following groups

— $\mathrm{Symp}(M)$ is the group of all C^∞-smooth *compactly supported* symplectomorphisms of M. If the manifold M itself is compact, then $\mathrm{Symp}(M)$ is the group of all symplectomorphisms of M.

— $\mathrm{SSymp}(M)$ is the connected component of $\mathrm{Symp}(M)$ containing unit e.

— $\mathrm{SSymp}^\sim(M)$ is the *universal covering group* of $\mathrm{SSymp}(M)$.

— More generally, we denote the universal covering of any connceted group G by $G^\sim$

— $\mathrm{Map}(M)$ is the *mapping class group* $\mathrm{Symp}(M)/\mathrm{SSymp}(M)$.

We have a natural topology on $\mathrm{Symp}(M)$ and hence we have a natural Borel structure on $\mathrm{Symp}(M)$. In particular, we have a notion of a *measurable function* on $\mathrm{Symp}(M)$. Let F be a measurable function on $\mathrm{Symp}(M)$, let $U\subset\mathbb{R}^N$ be an open domain and $\psi : U \mapsto \mathrm{Symp}(M)$ be a smooth map. Then $F\circ\psi$ is a measurable function on U in the usual sense.

1.3. Central extension of the group of symplectomorphisms. Equip the space $\mathbb{R}^{2n}$ with the standard symplectic structure. Consider an open set $\Omega\subset\mathbb{R}^{2n}$ and a symplectic embedding $\iota:\Omega\to M$ such that the measure of $M\setminus\iota(\Omega)$ is zero.

Remarks. a) We admit disconnected sets Ω.

b) It is pleasant (but not necessary) to think that $M\setminus\iota(\Omega)$ is a union of submanifolds.

Any element $g\in\mathrm{Symp}(M)$ induces a transformation $\iota^{-1}g\iota$ of Ω defined almost everywhere. Denote the group of all such transformations by $\mathrm{Symp}(M,\Omega,\iota)$. By the definition, $\mathrm{Symp}(M,\Omega,\iota)\simeq\mathrm{Symp}(M)$. This group contains $\mathrm{Symp}(\Omega)$ as a proper subgroup.

For $q\in\mathrm{Symp}(M,\Omega,\iota)$ and $x\in\Omega$, we denote by $q'(x)$ its Jacobi matrix at the point x. We define the 2-cocycle $C(q_1,q_2)$, where q_1, $q_2\in\mathrm{Symp}(M,\Omega,\iota)$,

[2]In conciderations of Subsection 4.2, this formula arises in a natural way

by the formula[3]

$$C(q_1,q_2) =$$
$$= \int_\Omega \mathrm{Im\ tr\ ln}\Big\{\Phi^{-1}\big[q_1'(q_2(m)\big]\ \Phi\big[q_1'(q_2(m))q_2'(m)\big]\ \Phi^{-1}\big[q_2'(m)\big]\Big\}dm =$$
$$\int_\Omega c\big(q_1'(q_2(m)), q_2'(m)\big)\, dm \quad (1.5)$$

Theorem 1.1 *a) The expression $C(q_1,q_2)$ defines an element of the second cohomology group $H^2(\mathrm{Symp}(M),\mathbb{R})$. In another words, the space $\mathrm{Symp}(M)\times\mathbb{R}$ with the product*

$$(q_1,x_1)\times(q_2,x_2) = (q_1\circ q_2, x_1+x_2+C(q_1,q_2)) \tag{1.6}$$

is a group.

b) This central extension of $\mathrm{Symp}(M)$ does not depend on a choice of the domain Ω and the map ι.

1.4. Triviality results for the cocycle C. We have a map from the universal covering group $\mathrm{SSymp}^\sim(M)$ to $\mathrm{SSymp}(M)$ and hence we can consider the cocycle C as a cocycle on $\mathrm{SSymp}^\sim(M)$.

Proposition 1.2 *Let $\Xi\subset\mathbb{R}^{2n}$ be an open domain. Then the central extension of $\mathrm{SSymp}^\sim(\Xi)$ defined by the cocycle C is trivial.*

Let M be a symplectic manifold. Consider an almost complex structure on the tangent bundle of M compatible with the symplectic structure (in particular, we obtain n-dimensional complex vector bundle). Assume that the corresponding Hermitian metric is positive definite.

For the complex bundle obtained in this way, consider its n-th exterior power L.

Proposition 1.3 *If the complex line bundle L on M is trivial, then the central extension of $\mathrm{SSymp}^\sim(M)$ defined by the cocycle C is trivial.*

Corollary 1.4 *For a noncompact 2-dimensional surfaces $\mathcal{M}$ of a finite genus, our cocycle is trivial on $\mathrm{SSymp}(\mathcal{M})$.*

Indeed, in this case, the group $\mathrm{SSymp}(\mathcal{M})$ is contractible, see [10], hence $\mathrm{SSymp}^\sim(\mathcal{M}) = \mathrm{SSymp}(\mathcal{M})$. Also, in this case, the line bundle L is trivial. Thus, by Proposition 1.3, our central extension is trivial.

1.5. Nontriviality results.

[3] This cocycle is induced from a cocycle on the group of measurable currents, see below Subsection 2.4. Also, below we propose a coordinate-less description of the cocycle C, see Subsection 2.9.

Theorem 1.5 *For a two-dimensional oriented (compact or noncompact) surface* $\mathcal{M}_g$ *of genus* $g \geq 3$*, our central extension of* $\mathrm{Symp}(\mathcal{M}_g)$ *is nontrivial in measurable cohomologies.*

Further, consider the standard lattice $\mathbb{Z}^{2n}$ in the standard symplectic space $\mathbb{R}^{2n}$. Consider the torus $\mathbb{T}^{2n} := \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. Denote by $\mathrm{Sp}(2n, \mathbb{Z})$ the subgroup in $\mathrm{Sp}(2n, \mathbb{R})$ consisting of all matrices (1.1) with integer elements.

The action of $\mathrm{Sp}(2n, \mathbb{Z})$ on $\mathbb{R}^{2n}$ induces the symplectic action of $\mathrm{Sp}(2n, \mathbb{Z})$ on $\mathbb{T}^{2n}$.

Theorem 1.6 *The central extension of* $\mathrm{Symp}(\mathbb{T}^{2n})$ *defined by the cocycle* C *is nontrivial for* $n \geq 3$.

In fact, we prove that this extension is nontrivial on the countable subgroup $\mathrm{Sp}(2n, \mathbb{Z}) \subset \mathrm{Symp}(\mathbb{T}^{2n})$. The latter statement is a kind of a rigidity theorem for lattices.

1.6. Some discussion.

a) Central extensions of of $\mathrm{SSymp}(M)$ were discussed in several works, see Kostant [9], Brylinski [6], Ismagilov [16], [17], Haller, Vizman [14]. Our construction differs from these constructions.

b) Consider a surface $\mathcal{M}_g$ of genus ≥ 3. It seems that our central extension in this case must be related to the Harer central extension [15] of the mapping class group $\mathrm{Map}(\mathcal{M}_g)$, see also [11].

But this relation now is not clear. Indeed, $\mathrm{Symp}(\mathcal{M}_g)$ is not a semidirect product $\mathrm{Map}(\mathcal{M}_g) \ltimes \mathrm{SSymp}(\mathcal{M}_g)$ and hence our construction does not induce automatically an extension of $\mathrm{Map}(\mathcal{M}_g)$.

Second, the central extension of $\mathrm{Map}(\mathcal{M}_g)$ induces a central extension of $\mathrm{Symp}(\mathcal{M}_g)$. Unfortunately, I do not know, gives our construction the same result or not.

c) Symplectic mapping class groups in higher dimensions were discussed by Seidel, see [27]; for results and references on topology of groups of symplectomorphisms, see surveys [20], [21].

d) For a two-dimensional surface $\mathcal{M}_g$, our central extension can be realized in a unitary representation of $\mathrm{Symp}(\mathcal{M}_g)$, this construction was obtained in [24]. I do not believe that a realization in a unutary representation is possible for dimensions ≥ 4 (realizations in nonunitary representations exist).

1.7. Structure of the paper. Details of construction of the cocycle C and proofs of Theorem 1.1 and Propositions 1.2–1.3 are contained in Section 2.

Theorems 1.5 and 1.6 are proved in Sections 3 and 4 respectively.

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2 Constructions of cocycles

2.1. Central extensions. Preliminaries. A. Let G be a group, let A be an Abelian group (in our work, A is the additive group of $\mathbb{R}$, $\mathbb{Z}$, or the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, we write the operation in A in the additive form). A *central extension* of G by the group A (or A-extension of G) is a group $\widetilde{G}$ such that A is a central subgroup in $\widetilde{G}$ and $\widetilde{G}/A \simeq G$.

The set $\widetilde{G}$ can be identified noncanonically with the product $G \times A$ and the homomorphism $\widetilde{G} \to G$ can be identified with the projection $G \times A \to G$. Then the multiplication in $G \times A$ must have the form

$$(g_1, a_1) \cdot (g_2, a_2) = \bigl(g_1 g_2, a_1 + a_2 + c(g_1, g_2)\bigr) \tag{2.1}$$

where the function $c : G \times G \to A$ (a *2-cocycle*) satisfies the identities

$$c(e, g) = c(g, e) = 0 \qquad \text{for all } g\ ; \tag{2.2}$$

$$c(g_1, g_2) + c(g_1 g_2, g_3) = c(g_1, g_2 g_3) + c(g_2, g_3) \tag{2.3}$$

where e is the unit of G. The first condition means that $(e, 0)$ is the unit of $\widetilde{G}$. The second condition is equivalent to the associativity of the multiplication (2.1).

B. If we change the identification $G \times A$ and $\widetilde{G}$, then $c(g_1, g_2)$ is changed according the rule

$$c(g_1, g_2) \mapsto c(g_1, g_2) - \gamma(g_1) - \gamma(g_2) + \gamma(g_1 g_2) \tag{2.4}$$

where $\gamma : G \to A$ is some function such that $\gamma(e) = 0$.

The central extension is *trivial* if $c(g_1, g_2)$ can be transformed to 0 by the operation (2.4), i.e.,

$$c(g_1, g_2) = \gamma(g_1) + \gamma(g_2) - \gamma(g_1 g_2) \tag{2.5}$$

In this case $\widetilde{G} = G \times A$ and we say that $\gamma(g)$ is a *trivializer* of c.

C. If γ, γ' are two trivializers of the same cocycle c, then $\gamma - \gamma'$ is a homomorphism $G \to A$.

D. The additive group of functions $c(g_1, g_2)$ satisfying (2.2)–(2.3) is denoted by $C^2(G, A)$ (*group of cocycles*); the group of functions having the form (2.5) is denoted by $B^2(G, A)$ (*the group of coboundaries*). The *second cohomology group* is the factor-group

$$H^2(G, A) := C^2(G, A)/B^2(G, A)$$

E. Let B be an Abelian group, A be its subgroup, $C = B/A$ be the factor-group. Then we have the obvious maps

$$C^2(G, A) \to C^2(G, B) \to C^2(G, C); \tag{2.6}$$

The first map means that an A-valued function c is also a B-valued function; considering a composition of a function $G \times G \to B$ and the homomorphism $B \to C$, we obtain the second map. We also have the corresponding map in cohomologies

$$H^2(G, A) \to H^2(G, B) \to H^2(G, C) \tag{2.7}$$

F. Let G, G' be groups, let $\theta : G' \to G$ be a homomorphism. Then θ induces a natural map $C^2(G, A) \to C^2(G', A)$, i.e., for a cocycle $c \in C^2(G, A)$ we consider the cocycle $c(\theta(g_1'), \theta(g_2')) \in C^2(G', A)$. Hence we also have a map of cohomologies

$$H^2(G, A) \to H^2(G', A)$$

G. Let G be a group, fix $h \in G$.

The cocycle $c(h^{-1}g_1h, h^{-1}g_2h)$ is equivalent to $c(g_1, g_2)$, see [5], III.8.1.

H. Now let G, A be topological groups. We say that a cocycle $c \in H^2(G, A)$ is nontrivial in measurable cohomologies if it can not be trivialized (see (2.6)) by a measurable trivializer γ.

2.2. A model of $\mathrm{Sp}(2n, \mathbb{R})$**.** In 1.1, we realized the group $\mathrm{Sp}(2n, \mathbb{R})$ as the group of complex matrices having the block structure

$$g = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} \tag{2.8}$$

preserving the skew-symmetric bilinear form K with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Then the matrix (2.8) also

— preserves the indefinite Hermitian form M with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e.,

$$g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g^* = g^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.9}$$

– preserves the real subspace $V \subset \mathbb{C}^n$, consisting of the vectors $(h, \overline{h})$, moreover, matrices (2.8) commute with the antilinear operator $(p, q) \mapsto (\overline{q}, \overline{p})$.

Now let us give a coordinateless description of this realization of $\mathrm{Sp}(2n, \mathbb{R})$. Consider an n-dimensional complex space V equipped with a positive definite Hermitian form $H(\cdot, \cdot)$. Denote by $V_\mathbb{R}$ the same space considered as $2n$-dimensional linear space over $\mathbb{R}$. The operator $v \mapsto iv$ in V is also a linear operator in $V_\mathbb{R}$, we denote it by I.

Denote by $\{\cdot, \cdot\}$ the imaginary part of the Hermitian form H, it is a skew-symmetric bilinear form on $V_\mathbb{R}$. The group preserving this form is $\mathrm{Sp}(2n, \mathbb{R})$.

Consider the complexification $(V_\mathbb{R})_\mathbb{C}$ of the space $V_\mathbb{R}$. It is a $2n$-dimensional complex linear space equipped with several additional structures

1) We have an operator I such that $I^2 = -1$.

2) Since $I^2 = -1$, the eigenvalues of I are $\pm i$. Denote by $V_\pm$ the corresponding eigenspaces. Then $V = V_+ \oplus V_-$.

3) We have the operation Q of complex conjugation $Q : v + iw \mapsto v - iw$, where $v, w \in V_{\mathbb{R}}$. It satisfies $QV_{\pm} = V_{\mp}$

Now we consider the action of $\mathrm{Sp}(2n, \mathbb{R})$ in $(V_{\mathbb{R}})_{\mathbb{C}}$. An operator $g \in \mathrm{Sp}(2n, \mathbb{R})$ preserves the bilinear form in $(V_{\mathbb{R}})_{\mathbb{C}}$ and commutes with the complex conjugation.

Representing it as a block operator $V_+ \oplus V_- \to V_+ \oplus V_-$, we obtain the block matrix representation (2.8)

The Hermitian form M on $(V_{\mathbb{R}})_{\mathbb{C}}$ is

$$M(v + iw, v' + iw') = \{v, w'\} - \{w, v'\} + i\{v, v'\} + i\{w, w'\}$$

it is more natural to say that we extend $i\{\cdot, \cdot\}$ as an Hermitian form from $V_{\mathbb{R}}$ to $(V_{\mathbb{R}})_{\mathbb{C}}$.

2.3. The Berezin cocycle on $\mathrm{Sp}(2n, \mathbb{R})$**.**

Theorem 2.1 *a) The function* $\mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{Sp}(2n, \mathbb{R}) \to \mathbb{R}$ *given by*

$$c(g_1, g_2) = \mathrm{Im}\ \mathrm{tr}\ \ln\Big[\Phi(g_1)^{-1}\Phi(g_1 g_2)(\Phi(g_2)^{-1}\Big] \tag{2.10}$$

is well-defined.

b) The function $c(g_1, g_2)$ *is a 2-cocycle.*

c) The $\mathbb{R}$*-valued cocycle* $\frac{1}{2\pi} c(g_1, g_2)$ *can be reduced to a* $\mathbb{Z}$*-valued cocycle.*[4] *The corresponding* $\mathbb{Z}$*-extension of* $\mathrm{Sp}(2n, \mathbb{R})$ *coincides with the universal covering group of* $\mathrm{Sp}(2n, \mathbb{R})$*.*

d) *The cocycle* c *is uniformly bounded on* $\mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{Sp}(2n, \mathbb{R})$

$$|c(g_1, g_2)| < n\pi/2 \tag{2.11}$$

Remark. In [3], Berezin wrote the following $\mathbb{T}$-cocycle on $\mathrm{Sp}(2n, \mathbb{R})$ and on its infinite-dimensional analogue

$$\sigma(g_1, g_2) = \det(1 + \Phi(g_1)^{-1}\Psi(g_1) \cdot \overline{\Psi}(g_2)\Phi(g_2)^{-1})^{-1/2}$$

Trivial calculation shows that

$$\sigma(g_1, g_2) = \exp\{-c(g_1, g_2)/2\}$$

This cocycle can be trivialized on the two-sheet covering of $\mathrm{Sp}(2n, \mathbb{R})$. For $n = 1$, an explicit formula for $\mathbb{R}$-valued cocycle c was written by Guichardet [13].

Proof. a) We have

$$\Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2) + \Psi(g_1)\overline{\Psi}(g_2)$$

Hence

$$\Phi(g_1)^{-1}\Phi(g_1 g_2)\Phi(g_2)^{-1} = 1 + \Phi(g_1)^{-1}\Psi(g_1) \cdot \overline{\Psi}(g_2)\Phi(g_2)^{-1} \tag{2.12}$$

[4] I.e., c is contained in the image of the map $H^2(\mathrm{Sp}(2n, \mathbb{R}), \mathbb{Z}) \to H^2(\mathrm{Sp}(2n, \mathbb{R}), \mathbb{R})$.

Equations (2.9) imply

$$\Phi\Phi^* - \Psi\Psi^* = 1; \qquad \Phi^*\Phi - \Psi^t\overline{\Psi} = 1 \tag{2.13}$$

Hence Φ is invertible and we have the following inequalities for norms

$$\|\Phi^{-1}\Psi\| < 1, \qquad \|\overline{\Psi}\Phi^{-1}\| < 1 \tag{2.14}$$

Thus,

$$\|\Phi(g_1)^{-1}\Psi(g_1)\cdot\overline{\Psi}(g_2)\Phi(g_2)^{-1}\| < 1$$

We define the logarithm of (2.12) by

$$\ln(1+Z) := \sum_{n=1}^{\infty}(-1)^{n+1}Z^n/n$$

Since the norm of our Z is < 1, our series converges.

b) Let A, B be $n\times n$ matrices and $\|A-1\|$, $\|B-1\|$ be sufficiently small. Then

$$\mathrm{tr}\ \ln(AB) = \mathrm{tr}\ \ln A + \mathrm{tr}\ \ln B$$

Hence, for g_1, g_2 lying in a small neighborhood of the unit we can write

$$\mathrm{tr}\ \ln\Big[\Phi(g_1)^{-1}\Phi(g_1g_2)\Phi(g_2)^{-1}\Big] = -\mathrm{tr}\ \ln\Phi(g_1) + \mathrm{tr}\ \ln\Phi(g_1g_2) - \mathrm{tr}\ \ln\Phi(g_2)$$

Now the cocycle identity (2.3) became trivial for g_1, g_2, g_3 lying in a sufficiently small neighborhood of unit.

But all our expressions are real analytic and the group $\mathrm{Sp}(2n,\mathbb{R})$ is connected. Hence the cocycle identity (2.3) is valid for all g_1, g_2, $g_3 \in \mathrm{Sp}(2n,\mathbb{R})$.

A verifying of (2.2) is trivial.

c) As we have seen $\det\Phi \neq 0$. Assume

$$\gamma(g) := \mathrm{Im}\ \mathrm{tr}\ \ln\Phi(g) = \arg\ln\det\Phi(g) \tag{2.15}$$

where $0 \le \arg z < 2\pi$. Obviously,

$$\alpha := \frac{1}{2\pi}\Big(c(g_1,g_2) + \gamma(g_1) + \gamma(g_2) - \gamma(g_1g_2)\Big) \in \mathbb{Z}$$

since $\exp(2\pi\alpha) = 1$, and the first statement is proved.

The $\mathbb{Z}$-central extension of $\mathrm{Sp}(2n,\mathbb{R})$ obtained in this way can be considered as the subset $\mathrm{Sp}(2n,\mathbb{R})^\sim \subset \mathrm{Sp}(2n,\mathbb{R})\times\mathbb{R}$ consisting of pairs

$$(g,x), \qquad \text{where } g\in\mathrm{Sp}(2n,\mathbb{R}),\ x\in\mathbb{R} \text{ and } \det\Phi(g)/|\det\Phi(g)| = e^{ix},$$

the multiplication is given by (2.1). Obviously, the projection $\mathrm{Sp}(2n,\mathbb{R})^\sim \to \mathrm{Sp}(2n,\mathbb{R})$ is a covering map. For $y\in\mathbb{R}$ consider the matrix $g(y)$ with $\Psi(y) = 0$ and Φ being the diagonal matrix with entries e^{iy}, 1,…, 1. The map $y \mapsto$

$(g(y), y)$ is a continuous map $\mathbb{R} \to \mathrm{Sp}(2n, \mathbb{R})^{\sim}$ and hence $\mathrm{Sp}(2n, \mathbb{R})^{\sim}$ is connected. Thus $\mathrm{Sp}(2n, \mathbb{R})^{\sim}$ is a covering group for $\mathrm{Sp}(2n, \mathbb{R})$.

It remains to notice that the unitary group $\mathrm{U}(n)$ is a deformation retract of $\mathrm{Sp}(2n, \mathbb{R})$ and the loop $g(y)$, $y \in [0, 2\pi]$, is a generator of the fundamental group $\pi_1(\mathrm{U}(n))$. Hence $\mathrm{Sp}(2n, \mathbb{R})^{\sim}$ is a universal covering of $\mathrm{Sp}(2n, \mathbb{R})$.

d) Let Z be a matrix satisfying $\|Z\| < 1$. Let $\lambda_j(Z)$ be its eigenvalues. Obviously, $|\lambda_j| < 1$. Let us prove that

$$\mathrm{tr}\ \ln(1 + Z) = \sum \ln(1 + \lambda_j(Z)) \tag{2.16}$$

For a self-adjoint matrix Z this identity is obvious. The both sides of (2.16) are complex analytic in Z and hence this identity is valid if $\|Z\| < 1$.

But $|\lambda_j(Z)| < 1$ implies $|\mathrm{Im}\ (1 + \lambda_j)| < \pi/2$ and we obtain (2.11). $\square$

Remark. Obviously, the expression

$$\mathrm{Re}\ \mathrm{tr}\ \ln\Big[\Phi(g_1)^{-1}\Phi(g_1 g_2)(\Phi(g_2)^{-1}\Big]$$

also satisfies the cocycle equation (2.3), but this cocycle is trivial, since its trivializer

$$\gamma(g) = \mathrm{Re}\ \mathrm{tr}\ \ln \Phi(g) = \ln|\det(\Phi(g))|$$

is well defined (but similar expression (2.15) is multivalued).

2.4. Groups $\mathfrak{B}(X, G)$. Denote by X some Lebesgue space with a continuous measure[5] μ. For example, we can consider X being an arbitrary symplectic manifold. Denote by $\mathrm{Ams}(X)$ the group of all measure-preserving maps from the space X to itself (by definition, these maps are defined almost everywhere).

Let G be an arbitrary group (below $G = \mathrm{Sp}(2n, \mathbb{R})$). Denote by $\mathcal{F}(X, G)$ the group of all measurable functions $f : X \to G$ such that

$$f(x) = 1 \qquad \text{outside a set of finite measure}$$

If X itself has a finite measure, then we can forget the last condition.

Consider the semidirect product

$$\mathfrak{B}(X, G) = \mathrm{Ams}(X) \ltimes \mathcal{F}(X, G)$$

Elements of the group $\mathfrak{B}(X, G)$ are pairs $\{p, h\} \in \mathrm{Ams} \times \mathcal{F}(X, G)$ and the product is given by

$$\{p_1(x), h_1(x)\} * \{p_2(x), h_2(x)\} := \{p_1(p_2(x)), h_1(p_2(x))h_2(x)\}$$

Remark. Let the group G acts on a manifold Y by transformations $y \mapsto yg$. Consider the space $\mathcal{L}(X, Y)$ of all Y-valued measurable functions on X. The group $\mathfrak{B}(X, G)$ acts in this space by the transformations

$$T(p, h)f(x) = f(p(x))h(x) \tag{2.17}$$

[5] In fact, any such space is equivalent to a segment $[a, b] \subset \mathbb{R}$ equipped with the Lebesgue measure or to the whole line $\mathbb{R}$.

2.5. A central extension of $\mathfrak{B}(X,G)$. Let $c \in H^2(G,\mathbb{R})$ be a bounded cocycle. We define the function $C : \mathfrak{B}(X,G) \times \mathfrak{B}(X,G) \to \mathbb{R}$ by

$$C(\{p_1,h_1\},\{p_2,h_2\} = \int_X c\big(h_1 \circ p_2(x), h_2(x)\big)\, d\mu(x) \tag{2.18}$$

Theorem 2.2 *The function C is a 2-cocycle on $\mathfrak{B}(X,G)$.*

We denote the corresponding central extension of $\mathfrak{B}(X,G)$ by $\widetilde{\mathfrak{B}}(X,G)$.

Proof. We directly verify the cocycle identity (2.3). First,

$$C\Big(\{p_1,h_1\}\,,\,\{p_2,h_2\}\Big) + C\Big(\{p_1,h_1\} * \{p_2,h_2\}\,,\,\{p_3,h_3\}\Big) = \tag{2.19}$$

$$= \int_X c\Big([h_1 \circ p_2](x)\ ,\ h_2(x)\Big)\, d\mu(x) + \tag{2.20}$$

$$+ \int_X c\Big([h_1 \circ p_2 \circ p_3](x) \cdot [h_2 \circ p_3](x)\ ,\ h_3(x)\Big)\, d\mu(x)$$

Secondly,

$$C\Big(\{p_2\,,\,h_2\}\,,\,\{p_3\,,\,h_3\}\Big) + C\Big(\{p_1,h_1\}\,,\,\{p_2,h_2\} * \{p_3,h_3\}\Big) = \tag{2.21}$$

$$= \int_X c\Big([h_2 \circ p_3](x)\ ,\ h_3(x)\Big)\, d\mu(x) + \tag{2.22}$$

$$+ \int_X c\Big([h_1 \circ p_2 \circ p_3](x)\ ,\ [h_2 \circ p_3](x) \cdot h_3(x)\Big)\, d\mu(x)$$

We substitute $x = p_3(y)$ to (2.20) and replace the summand (2.20) by

$$\int_X c\Big([h_1 \circ p_2 \circ p_3](x)\ ,\ [h_2 \circ p_3](x)\Big)\, d\mu(x)$$

We obtain that (2.19) equals (2.21) due the cocycle identity (2.3) for $c(\cdot,\cdot)$ with

$$g_1 = h_1 \circ p_2 \circ p_3(x), \quad g_2 = h_2 \circ p_3(x), \quad g_3 = h_3(x)$$

Remark. To apply this construction, we need in a group G, having a nontrivial $\mathbb{R}$-central extension. Some reasonable examples are $G = \mathbb{R}/\mathbb{Z}$, $\mathrm{U}(p,q)$, $\mathrm{SO}^*(2n)$, $\mathrm{SO}(n,2)$, and the group $\mathrm{SDiff}(S^1)$ of orientation preserving diffeomorphisms of the circle.

2.6. Another explanation of the central extension of $\mathfrak{B}(X,G)$. Denote by $\widetilde{G}$ the central extension of G defined by the cocycle c, by definition $\widetilde{G} \supset \mathbb{R}$. Concider the group $\mathfrak{B}(X,\widetilde{G})$ and its central Abelian subgroup $\mathcal{F}(X,\mathbb{R})$. Denote by Q the subgroup of $\mathcal{F}(X,\mathbb{R})$ consisting of functions f such that

$$\int_X f(x)\, d\mu(x) = 0$$

Obviously, Q is a normal subgroup in $\mathfrak{B}(X, \widetilde{G})$. It can readily checked that

$$\widetilde{\mathfrak{B}}(X, G) = \mathfrak{B}(X, \widetilde{G})/Q$$

2.7. Embedding $\mathrm{Symp}(M) \to \mathfrak{B}(M, \mathrm{Sp}(2n, \mathbb{R}))$**.** Now let $X \simeq M$ be a $2n$-dimensional symplectic manifold, let $\Omega \subset \mathbb{R}^{2n}$ be an open domain, and let $\iota : \Omega \to M$ be a symplectic embedding such that the set $M \setminus \iota(\Omega)$ has a zero measure. For $g \in \mathrm{Symp}(M)$, consider the map $q := \iota^{-1} g \iota : \Omega \to \Omega$ defined almost sure. Denote by $q'(x)$ the Jacobi matrix of g at point x.

The map

$$g \mapsto (q, q')$$

is an embedding $\mathrm{Symp}(M) \to \mathfrak{B}(M, \mathrm{Sp}(2n, \mathbb{R}))$. Our cocycle (1.5) on Symp is induced from the cocycle (2.18) on $\mathfrak{B}(M, \mathrm{Sp}(2n, \mathbb{R})$.

2.8. Groups of automorphisms $\mathrm{Aut}(R, M)$ **of bundles.** Now let $X = M$ be an m-dimensional manifold equipped with a volume form Ω. Consider an $2n$-dimensional real vector bundle $R \to M$ on M with fibers R_x, $x \in M$. Assume that fibers R_x are equipped with skew-symmetric bilinear forms $\{\cdot, \cdot\}_x$.

Denote by $\mathrm{Aut}(R, M)$ the group of smooth maps $\Theta : R \to R$ satisfying the conditions

1. An image of any fiber R_x is some fiber $R_{\theta(x)}$, and the map Θ induces a linear map R_x to $R_{\theta(x)}$ preserving the skew-symmetric form, i.e.,

$$\{\Theta v, \Theta w\}_{\xi(x)} = \{v, w\}_x$$

where v, $w \in R_x$.

2. The map $x \mapsto \theta(x)$ is a diffeomorphism of the base M preserving the volume form Ω.

3. For a noncompact manifold M, we have an additional conditions: there is a compact subset $L \subset M$ such that for any $x \notin L$ we have $\theta(x) = x$ and $\Theta v = v$ for any $v \in \mathbb{R}_x$.

The group $\mathrm{Aut}(R, M)$ admits an obvious embedding to the group $\mathfrak{B}(M, \mathrm{Sp}(2n, \mathbb{R}))$. Indeed, consider an arbitrary trivialization of the bundle R over an arbitrary open set $X \subset M$ of a complete measure. Then elements of $\mathrm{Aut}(R, M)$ induce transformations of $\mathfrak{L}(X, \mathbb{R}^{2n})$ of the type (2.17).

The embedding $\mathrm{Aut}(R, M) \to \mathfrak{B}(M, \mathrm{Sp}(2n, \mathbb{R}))$ described above is not canonical, but any two embeddings η_1, η_2 of this type are conjugated by some element $r \in \mathfrak{B}(M, \mathrm{Sp}(2n, \mathbb{R}))$:

$$\eta_2(\Theta) = r^{-1} \eta_1(\Theta) r \tag{2.23}$$

Thus, the central extension of $\mathfrak{B}(M, \mathrm{Sp}(2n, \mathbb{R})$ induces a central extension of the group $\mathrm{Aut}(R, M)$. The formula for the cocycle depends on the set $X \subset M$ and on a trivialization of the bundle. But due (2.23) all these cocycles are equivalent. Thus, our central extension is canonical.

2.9. A geometric construction of the central extension of $\mathrm{Aut}(R, M)$**.** Let R, M be the same as above.

Consider an almost complex structure on the bundle R. Recall that this is an operator $J_x : R_x \to R_x$ depending on x smoothly and satisfying

$$J_x^2 = -1; \qquad \{J_x v, J_x w\}_x = \{v, w\}_x$$

We also assume that the symmetric bilinear form

$$S_x(v, w) = \{J_x v, w\}_x$$

is positive definite on each fiber. Such almost complex structures exist, see, for instance, [22] (evidently, such structure is not unique). In particular, the tangent bundle TM became a complex n-dimensional bundle, but in this moment we prefer to consider TM is a real bundle with an additional structure.

Consider the fiber-wise complexification $R^{\mathbb{C}}$ of the vector bundle $R \to M$. In each fiber $(R_x)^{\mathbb{C}}$ of $R^{\mathbb{C}}$, we have two canonically defined subspaces

$$R_x^{\pm} := \ker(J_x \mp i), \qquad (R_x)^{\mathbb{C}} = R_x^+ \oplus R_x^-$$

Thus we can represent any real symplectic linear operator $h : R_x \to R_y$ as a complex block operator

$$h : R_x^+ \oplus R_x^- \to R_y^+ \oplus R_y^-$$

We denote by $\Phi(h)$ the block corresponding $R_x^+ \to R_y^+$.

The formula for the 2-cocycle is

$$C(\Theta^{(1)}, \Theta^{(2)}) = \int_M c\Big((\Theta^{(1)}\Big|_{R_{\theta^{(2)}(x)}}, \Theta^{(2)}\Big|_{R_x}\Big)\Omega(x)$$

2.10. Groups of symplectomorphisms. Now let M be a $2n$-dimensional symplectic manifold. Denote by R the tangent bundle to M. We have the natural embedding

$$\mathrm{Symp}(M) \subset \mathrm{Aut}(R, M)$$

and hence we can induce a 2-cocycle from the group $\mathrm{Aut}(R, M)$.

We also can put $\mathrm{Symp}(M)$ to $\mathfrak{B}(M, \mathrm{Sp}(2n, \mathbb{R}))$ directly, as it was done above in 1.3.

The cohomological class of the induced cocycle do not depend on the embeddings since all these embeddings are conjugated by interior automorphisms of $\mathfrak{B}(M, \mathrm{Sp}(2n, \mathbb{R}))$.

2.11. Proof of Proposition 1.2. Let Ξ be a domain in $\mathbb{R}^{2n}$. Then

$$\nu : (g, x) = \det \Phi(g'(x))/|\det \Phi(g'(x))| \tag{2.24}$$

is a well defined function

$$\mathrm{SSymp}(\Xi) \times \Xi \to \mathbb{T}$$

where $\mathbb{T}$ is the group of complex numbers z such that $|z| = 1$.

By the covering homotopy theorem, this map can be lifted to the map

$$\widetilde{\nu} : \mathrm{SSymp}^{\sim}(\Xi) \times \Xi \to \mathbb{R}$$

such that

$$\exp\bigl(i\widetilde{\nu}(g,x)\bigr) = \nu(g,x), \qquad \widetilde{\nu}(e,x) = 0$$

The function $\widetilde{\nu}$ is precisely a single-valued branch of

$$\mathrm{Im}\ \ln \det \Phi(g'(x)) = \ln\Bigl[\det \Phi(g'(x))/|\det \Phi(g'(x))|\Bigr]$$

Then

$$\Gamma^{\circ}(g) := \int_{\Xi} \widetilde{\nu}(g,x)\, dx \tag{2.25}$$

is a trivializer of the 2-cocycle $C(\cdot,\cdot)$ on $\mathrm{SSymp}(\Xi)$, i.e.

$$C(g_1,g_2) = \Gamma^{\circ}(g_1g_2) - \Gamma^{\circ}(g_1) - \Gamma^{\circ}(g_2)$$

Indeed, the right hand side is

$$\int \mathrm{Im}\ \ln \Phi\bigl([g_1'\circ g_2](x)\cdot g_2'(x)\bigr)\, dx - \int \mathrm{Im}\ \ln \Phi(g_1'(x))\, dx - \int \mathrm{Im}\ \ln \Phi(g_2'(x))\, dx$$

We change the variable $x \mapsto g_2(x)$ in the second summand and transform the expression to the form

$$\int \mathrm{Im}\ \mathrm{tr}\ \ln\Bigl[\Phi\bigl([g_1'\circ g_2](x)\bigr)^{-1} \Phi\bigl([g_1'\circ g_2](x)\cdot g_2'(x)\bigr)\Phi\bigl(g_2'(x)^{-1}\bigr)\Bigr]\, dx$$

This proves Proposition 1.2.

Remark. Generally, this trivializer is not unique, since (see 2.1) there are nontrivial homomorphisms $\mathrm{SSymp}(\Xi)^{\sim} \to \mathbb{R}$, namely the flux homomorphisms and the Calabi invariant (see, for instance [1], [22], we discuss them below in a special case). By the Banyaga Theorem [1], this list exhaust all the homomorphisms from $\mathrm{SSymp}^{\sim}$ to Abelian groups.

Corollary 2.3 *Let $\Xi \subset \mathbb{R}^4$ be an open domain. Then our central extension of the (disconnected) group* $\mathrm{Symp}(\Xi)$ *is trivial.*

Indeed, the group $\mathrm{Symp}(\mathbb{R}^4)$ is connected and contractible (Gromov [12]). By Proposition 1.2, its central extension is trivial. But $\mathrm{Symp}(\Xi) \subset \mathrm{Symp}(\mathbb{R}^4)$.

2.12. Proof of Proposition 1.3. In the case of general symplectic manifolds M, the sense of the expression (2.24) is not clear, since an operator $\Phi(\cdot)$ maps one fiber of a bundle to another one and hence its determinant is not well defined.

Consider an Hermitian structure on the tangent bundle TM to M compatible with the symplectic form (as in 2.9). Then TM becomes a complex bundle,

denote it by $\mathcal{T}M$ (it is also isomorphic to the subbundle $R^+ \subset R^{\mathbb{C}}$ defined above). Consider its maximal exterior power $\wedge^n \mathcal{T}M$.

Assume that the bundle $\wedge^n \mathcal{T}M$ is trivial. Fix its trivialization. This precisely means that we have defined determinants of the operators $\Phi(\cdot)$ connecting different fibers.

Then we repeat our argument with a covering homotopy and obtain Proposition 1.3.

3 Two-dimensional surfaces

Here we prove nontriviality of the cocycle C in the case of a two-dimensional oriented surface $\mathcal{M}$ of genus $g \geq 3$. Evidently, in 2-dimensional case the group $\mathrm{SSymp}(\mathcal{M})$ coincides with the group of volume preserving and orientation preserving diffeomorphisms. Also, $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$.

We fix the following notation

— Λ is a simply connected domain in $\mathbb{R}^2$ equipped with the form

$$\omega = dx \wedge dy$$

It convenient to think that Λ is a standard disk.

— Δ is a multiconnected domain in $\mathbb{R}^2$. It is convenient to think, that Δ is a disk with $k > 0$ holes and the exterior boundary of Δ is the boundary of Λ.

— $\sigma(S)$ is the area of a domain $S \subset \mathbb{R}^2$.

The group $\mathrm{Symp}(\Lambda) = \mathrm{SSymp}(\Lambda)$ is connected and contractible (Smale). The groups $\mathrm{Symp}(\Delta)$ are disconnected (this is obvious). The identity component $\mathrm{SSymp}(\Delta)$ is contractible (Earle, Eells [10]).

Our main arguments for a proof are: continuity of the cocycle in the topology of convergence in measure, formula (2.25) for a trivializer in a flat case, a certain Dehn relation in the Teichmuller group, and the Banyaga Theorem.

3.1. Global angle of rotation. For $\psi \in \mathbb{R}$ consider the unit vector

$$v_\psi = \cos\psi e_1 + \sin\psi e_2$$

Denote by S^1 the set of all unit vectors in $\mathbb{R}^2$.

Let $q \in \mathrm{Symp}(\Lambda)$. Consider a point $x \in \Lambda$ and a unit vector v_ψ applied at this point. Consider the image $w = q'(x)v$ of v under the Jacobi matrix $q'(x)$. The normalized vector $w/\|w\|$ has a form v_φ. We assume

$$\mathrm{ang}(q, x, v) := \varphi - \psi$$

i.e., $\mathrm{ang}(\cdot)$ is the angle of turning of a vector under a diffeomorphism.

Lemma 3.1 *a) There exists a unique continuous function (global turning angle)*

$$\mathrm{Ang} : \mathrm{SSymp}(\Lambda) \times \Lambda \times S^1 \to \mathbb{R}$$

such that

1) The composition of $\mathrm{Ang}(\cdot)$ *and the map* $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ *is* $\mathrm{ang}(\cdot)$*.*
2) $\mathrm{Ang}(e, x, v) = 0$*.*
3) Let U *be a neighborhood of boundary of* Λ*, where* $q(x) = x$*. Then*

$$\mathrm{Ang}(q, x, v) = 0 \qquad \text{for all } x \in U \text{ and all } v \in S^1$$

b) The function Ang *satisfies the identity*

$$\mathrm{Ang}(q_1 \circ q_2, x, v) = \mathrm{Ang}(q_2, x, v) + \mathrm{Ang}(q_1,\, q_2(x),\, q_2'(x)v) \tag{3.1}$$

Proof. a) Consider a map

$$f : \mathrm{SSymp}(\Lambda) \times \Lambda \times S^1 \to S^1$$

defined in the following way:

$$f(q, x, v) = \frac{q'(x)v}{\|q'(x)v\|}$$

Next, we consider the covering map

$$\widetilde{f} : \mathrm{SSymp}(\Lambda) \times \Lambda \times \mathbb{R} \to \mathbb{R}; \qquad \widetilde{f}(e, x, \varphi) = \varphi$$

defined by the covering homotopy Theorem (recall that $\mathrm{SSymp}(\Lambda)$ is simply connected). Then

$$\mathrm{Ang}(q, x, v_\psi) = \widetilde{f}(q, x, \psi) - \psi$$

b) is obvious. □

Corollary 3.2 *The global turning angle is well defined on the (disconnected) group* $\mathrm{Symp}(\Delta)$*.*

Proof. Indeed, each compactly supported symplectomorphism of $\Delta \subset \Lambda$ is also a symplectomorphism of Λ. □

In particular, Lemma 3.1 gives the following geometrically visual way of evaluation of $\mathrm{Ang}(\cdot)$.

Lemma 3.3 *Let* $q \in \mathrm{SSymp}(\Lambda)$*. Consider a point* z *near the boundary of* Λ*, where* $q(z) = z$*. Let* $\ell(t)$ *be a smooth curve,* $\ell(0) = z$*,* $\ell(1) = x$*,* $\frac{d}{dt}\ell(1) = v$*. Then*

$$\mathrm{Ang}(q, x, v) = \left\{\textit{total turning of vector } \frac{d}{dt}(q(\ell(t)))\right\} - \left\{\textit{total turning of vector } \frac{d}{dt}\ell(t)\right\}$$

PROOF. Indeed, Ang$(\cdot)$ must be continuous along the curve

$$\xi(t) := (q, \ell(t), \dot{\ell}(t)) \in \mathrm{SSymp} \times \Lambda \times S^1 \qquad \square$$

Let $\Phi(h)$, where $h \in \mathrm{SL}(2, \mathbb{R})$, be the same as above. In our case $\Phi(h)$ is an element of $\mathbb{C}$ and $|\Phi(h)| \geq 1$. Let

$$h = SA, \tag{3.2}$$

be the polar decomposition of h, where A is a rotation by some angle θ and S is a contraction-dilatation with respect two orthogonal axes. Then

$$\Phi(h)/|\Phi(h)| = e^{i\theta} \tag{3.3}$$

Lemma 3.4 *Consider a continuous branch of the function*

$$\gamma^\circ(g, x) := \mathrm{Im}\ \ln \Phi(g'(x))$$

on $\mathrm{SSymp}(\Lambda) \times \Lambda)$ *such that* $\gamma(e, x) = 0$. *We have*

$$|\mathrm{Ang}(q, x, v) - \mathrm{Im}\ \ln \Phi(g'(x))| < \pi/2 \tag{3.4}$$

for all $q \in \mathrm{SSymp}(\Lambda)$, $x \in \Lambda$, $v \in S^1$.

PROOF. We can define the both functions in the following way. Consider the map

$$F : \mathrm{SSymp}(\Lambda) \times \Lambda \to \mathrm{SL}(2, \mathbb{R})$$

given by $(q, x) \mapsto q'(x)$. Consider the covering map

$$\widetilde{F} : \mathrm{SSymp}(\Lambda) \times \Lambda \to \mathrm{SL}(2, \mathbb{R})^\sim$$

Let h ranges in $\mathrm{SL}(2, \mathbb{R})$. The function $\alpha(h) := \mathrm{Im}\ \ln \Phi(h)$ is a function on $\mathrm{SL}(2, \mathbb{R})^\sim$. Also the $\mathbb{R}$-valued angle of turning of a unit vector $v \in \mathbb{R}^2$ under $h \in \mathrm{SL}(2, \mathbb{R})^\sim$ is a function on $\mathrm{SL}(2, \mathbb{R})^\sim$ (denote it by $\beta_v(h)$). We have

$$\mathrm{Ang}(g, x, v) = \beta_v(\widetilde{F}(g'(x))), \qquad \gamma^\circ(g, x) = \alpha(\widetilde{F}(g'(x))) \tag{3.5}$$

Hence it is sufficient to show that

$$|\alpha(h) - \beta_v(h)| < \pi/2 \tag{3.6}$$

This is a corollary of the following statement:

– Let $h \in \mathrm{SL}(2, \mathbb{R})$, let $h = SA$ be its polar decomposition (3.2). Then the angle between hv and Av is less than $\pi/2$.

The latter statement is obvious. Passing to the covering group $\mathrm{SL}(2, \mathbb{R})^\sim$, we obtain (3.6); applying (3.5), we obtain (3.4). $\square$

Lemma 3.5

$$\max_{x,v} |\mathrm{Ang}(q_1 \circ q_2, x, v)| \leq \max_{x,v} |\mathrm{Ang}(q_1, x, v)| + \max_{x,v} |\mathrm{Ang}(q_2, x, v)|$$

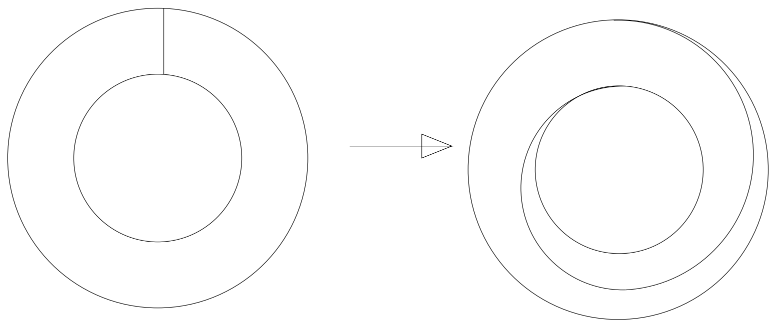

Figure 1: A standard twist.

Proof. This follows from (3.1).

3.2. Twists. Let $r > 0$, φ be the polar coordinates on the plane. Consider a ring $S : a \leq r \leq b$. A *standard right twist* is a diffeomorphism $q : S \to S$ that fix the both boundary circles, i.e., $q(a, \varphi) = (a, \varphi)$, $q(b, \varphi) = (b, \varphi)$, and

$$q(r, \varphi) = q(r, \varphi + \mu(r))$$

where μ is an arbitrary smooth increasing function on $[0, \infty)$ such that

$$\mu(x) = 0 \qquad \text{for } x < a + \delta; \qquad \mu(r) = 2\pi \qquad \text{for } x > b - \delta$$

for some $\delta > 0$, see Fig. 1.

Remark. The image of a right twist under the orientation preserving map $(r, \varphi) \mapsto (ab/r, -\varphi)$ is a right twist again. A diffeomorphism inverse to a right twist is not a right twist (it is called a *left twist*). $\square$

Consider a closed smooth non-self-intersecting curve C on the surface $\mathcal{M}$. Consider a 'small' neighborhood U of C. For some standard ring $S \subset \mathbb{R}^2$ consider some area-preserving and orientation preserving diffeomorphism $p : S \to U$. Consider a diffeomorphism $h \in \mathrm{Symp}(\mathcal{M})$ having the form

– $h(m) = m$ if $m \notin U$

– $p^{-1}hp$ is a standard right twist of S.

We call such diffeomorphism by a *twist about the curve* C supported by the neighborhood U.

A *Dehn twist* is a twist about a nonseparating curve B (i.e., the domain $\mathcal{M} \setminus B$ is connected). For any two nonseparating curves B_1, B_2 there is a diffeomorphism $q \in \mathrm{Symp}(\mathcal{M})$ such that $q(B_1) = B_2$ (see, for instance, [18]). If h is a right twist about B_1, then $q^{-1}hq$ is a right twist about B_2.

3.3. ε-twists. Below we use families of twists depending on parameters and we are need in some uniform estimates in parameters. By this reason, we give more rigid definitions.

Fix a function ν on $(-\infty, \infty)$ satisfying the conditions

– $\nu = 0$ on the ray $(-\infty, 0)$ and $\nu = 2\pi$ on the ray $x \geq 1/2$

– ν is C^∞-smooth and increasing.

This function remains fixed until the end of this section.

A *standard ε-twist* is a family $q(\varepsilon)$ of diffeomorphisms of the ring $1/2 < r < 3/2$ depending in a parameter ε, they have the form

$$\big(r,\varphi\big) \mapsto \big(r, \varphi + \nu\big(\tfrac{r-1}{\varepsilon}\big)\big), \qquad 0 < \varepsilon < 1/2$$

(in particular, this map is identical outside the ring $1 < r < 1 + \varepsilon/2$).

Now let C be a closed non-self-intersecting curve on $\mathcal{M}$. Let h be an embedding of some ring $1 - \delta < r < 1 + \delta$ to $\mathcal{M}$ such that the C is the image of the circle $r = 1$. We define an *ε-twist* about C as a family of diffeomorphisms of $\mathcal{M}$ depending on ε and given by

$$m \mapsto h \circ q(\varepsilon) \circ h^{-1}(m), \qquad \text{for } m \in U$$
$$m \mapsto m, \qquad \text{for } m \notin U$$

The parameter ε ranges in the interval $(0, \delta)$.

We fix some notation

– $\mathfrak{T}_C(\varepsilon)$ for an ε-twist about C; we omit h from our notation, but we remember that h is fixed.

– $\mathfrak{U}(\varepsilon) = \mathfrak{U}_C(\varepsilon)$ for the support of the twist $\mathfrak{T}_C(\varepsilon)$. The area of $\mathfrak{U}_C(\varepsilon)$ tends to 0 as $\varepsilon \to 0$, more precisely, $\sigma(\mathfrak{U}_C(\varepsilon)) = O(\varepsilon)$.

3.4. Trivializator of the cocycle C on a twist. The cocycle $C(\cdot,\cdot)$ on the group $\mathrm{SSymp}(\Lambda)$ is trivial, its canonical trivializer

$$\Gamma^\circ(q) = \int_\Lambda \mathrm{Im}\ \ln \Phi(q'(x))\, dx \tag{3.7}$$

was defined above in 2.11.

Lemma 3.6 *Let B be a smooth non self-intersecting curve in Λ surrounding the domain S. Then*

$$\Gamma^\circ(\mathfrak{T}_B(\varepsilon)) = 2\pi\sigma(S) + O(\varepsilon), \qquad \textit{as } \varepsilon \to 0$$

PROOF. The strip $\mathfrak{U}(\varepsilon)$ separates Λ into two domains, i.e., the exterior domain W^{ext} and the interior domain W^{int}. Let $\ell(t)$ be a (short) curve intersecting the strip $\mathfrak{U}(\varepsilon)$, let $\ell(0) \in W^{ext}$, $\ell(\delta) \in W^{int}$. The Jacobi matrix $[\mathfrak{T}'_B \circ \ell](t)$ is 1 at $t = 0$ and at $t = \delta$. But the curve $[\mathfrak{T}' \circ \ell](t)$ is noncontractible in $\mathrm{SL}(2, (\mathbb{R}))$ and moreover, it is a generator of the fundamental group $\pi_1(\mathrm{SL}_2(\mathbb{R}))$ (it is sufficient to verify that the vector $[\mathfrak{T}'_B \circ \ell](t)\dot{\ell}(t)$ is turning by the angle 2π as we pass $\ell(t)$).

Hence $\ln \Phi(\mathfrak{T}'_B(\ell(\delta)) = 2\pi$, and thus $\ln \Phi(\mathfrak{T}'_B(x)) = 2\pi$ for $x \in W^{int}$.

Further,

$$\Gamma^\circ(\mathfrak{T}_B(\varepsilon)) = \int_\Lambda \mathrm{Im}\ \ln \Phi(\mathfrak{T}'_B(x))\, dx = \int_{W^{ext}} + \int_{\mathfrak{U}(\varepsilon)} + \int_{W^{int}}$$

The integrand in the first summand is 0, the integrand in the last summand is 2π (and hence the integral over W^{int} is $2\pi\sigma(S)+O(\varepsilon)$). Thus it is sufficient to show that $\gamma^\circ = \mathrm{Im}\ \ln \Phi(\mathfrak{T}'_B(x))$ is bounded in the thin strip $\mathfrak{U}(\varepsilon)$. By Lemma 3.4, it is sufficient to show the boundedness of $\mathrm{Ang}(\mathfrak{T}_B, x, v)$, and by Lemma 3.3, the latter statement is obvious. $\square$.

3.5. Flux homomorphisms. Now, let $\Lambda \subset \mathbb{R}^2$ be a disk (or simply connected domain) bounded by a curve Z_0. Let $\Delta \subset \Lambda$ be a multi-connected domain bounded by smooth curves Z_0 (the exterior boundary) and $Z_1, \dots, Z_k$ (boundaries of the holes). Denote by ω the standard symplectic form $dx \wedge dy$ on $\mathbb{R}^2$. Let λ be a 1-form on $\mathbb{R}^2$ such that $d\lambda = \omega$ (for instanse, $\lambda = x\,dy$).

For each j, fix a non self-intersecting curve $u_j(t)$ connecting Z_0 and Z_j, let $u_j(0) \in Z_0$, $u_j(1) \in Z_j$. We define the function $\tau_j(g)$ in the variable $g \in \mathrm{Symp}(\Delta)$ by

$$\tau_j(g) = \int_{u_j} \lambda - \int_{u_j} g^*\lambda = \int_{u_j} \lambda - \int_{gu_j} \lambda = \int_{D_j} \omega \tag{3.8}$$

where D_j is a 2-cycle whose boundary is $u_j - gu_j$. Non-formally, $\tau_j(g)$ is the oriented area of the domain bounded by the curves $u_j(t)$ and $g(u_j(t))$.

Lemma 3.7 *a) τ_j does not depend on a choice of u_j.*
b) τ_j is a homomorphism $\mathrm{Symp}(\Delta) \to \mathbb{R}$.

The maps τ_j are called by *flux homomorphisms*, for definitions and properties in a general symplectic case, see Banyaga [1], McDuff, Salamon [22].

Proof. a) Let u'_j be another curve, let τ'_j be another map. Let R be a 2-cycle on $\mathbb{R}^2$ with boundary $u_j - u'_j$,

$$\tau_j(g) - \tau'_j(g) = \Bigl(\int_{u_j} \lambda - \int_{u'_j} \lambda)\Bigr) - \Bigl(\int_{gu_j} \lambda - \int_{gu'_j} \lambda\Bigr) = \int_R \omega - \int_{gR} \omega = 0$$

b)

$$\tau_j(g_1g_2) = \int_{u_j} \lambda - \int_{g_1g_2u_j} \lambda = \int_{u_j} \lambda - \int_{g_2u} \lambda + \int_{g_2u} \lambda - \int_{g_1(g_2u_j)} \lambda = \tau_j(g_2) + \tau_j(g_1)$$

Remark. In a general case, flux homomorphisms are defined on the connected group SSymp. In our case, we obtain homomorphisms τ_j of group $\mathrm{Symp}(\Delta)$, these homomorphisms depend on an embedding of the symplectic manifold Δ to $\mathbb{R}^2$, since the areas of the holes Z_j participate in formula (3.8)

3.6. Values of fluxes on twists. We preserve the notation of the previous subsection. The following statement is obvious.

Lemma 3.8 *Let C be a Jordan contour in Δ surrounding a domain S. Then*

$$\tau_j(\mathfrak{T}_C(\varepsilon)) = \begin{cases} \sigma(S) + O(\varepsilon), & \text{if } Z_j \subset S \\ 0, & \text{otherwise} \end{cases}$$

3.7. The Calabi homomorphism. We preserve the notation of the previous subsection. Consider a 1-form λ on $\mathbb{R}^2$ such that $d\lambda = \omega$.

For $g \in \mathrm{SSymp}(\Lambda)$, the form $g^*\lambda - \lambda$ is closed and hence it is exact.

Hence

$$g^*\lambda - \lambda = dF \tag{3.9}$$

The function F is defined up to an additive constant. We assume $F = 0$ on Z_0. Then

$$\varkappa(g) := \int_\Lambda F\,dx\,dy \tag{3.10}$$

is a homomorphism $\mathrm{SSymp}(\Lambda) \to \mathbb{R}$. It can easily be checked that $\varkappa(g)$ does not depend on a choice of a potential λ. The homomorphism $\varkappa(g)$ is called by the *Calabi invariant*.

Next, we restrict the Calabi invariant to the group $\mathrm{Symp}(\Delta) \subset \mathrm{SSymp}(\Lambda)$ and hence we obtain the homomorphisms $\mathrm{Symp}(\Delta) \to \mathbb{R}$.

Remark. In general situation, the Calabi invariant is defined on the kernel of all the flux homomorphisms $\subset$ Symp. In our case, it is defined globally, but its definition is not canonical, it depends on an embedding of the symplectic domain Δ to $\mathbb{R}^2$. Indeed, formula (3.10) includes integration over the holes, this operation is not invariantly defined on the manifold Δ. But these integrals over the holes give a linear combination of flux homomorphisms.

By the Banyaga Theorem (see [1], see also some useful additions in Rousseau [26], see also [2]), the intersection of the kernels of all the flux homomorphisms and of the kernel of the Calabi invariant is a perfect group (i.e., it has no homomorphisms to Abelian groups; moreover, this intersection is a simple group).

Thus, in our case, each measurable[6] homomorphism $\mathrm{SSymp}(\Delta) \to \mathbb{R}$ is a linear combination of k flux homomorphisms τ_j and of the Calabi invariant $\varkappa$.

3.8. Values of the Calabi invariants on twists.

Lemma 3.9 *Let B be a smooth non self-intersecting curve in Λ surrounding the domain S. Then*

$$\varkappa(\mathfrak{T}_B(\varepsilon)) = \sigma(S)^2 + O(\varepsilon), \qquad \varepsilon \to 0 \tag{3.11}$$

Proof. We preserve the notation from the proof of previous Lemma 3.6. Let us choose $\lambda = \frac{1}{2}(x\,dy - y\,dx)$. Let F be the function (3.9).

$$F(\ell(1)) = \int_\ell (\mathfrak{T}_B^*\lambda - \lambda) = \int_{\mathfrak{T}_B\ell} \lambda - \int_\ell \lambda$$

We obtain an integral over a closed curve Q composed from ℓ and $\mathfrak{T}_B\ell$.

This curve lies in the strip $\mathfrak{U}_B(\varepsilon)$ and surrounds W^{int}. By the Green formula, $F \simeq \sigma(S)$ in W^{int}.

[6]The Choice Axiom implies an "existence" of non-measurable homomoprhisms $\psi : \mathbb{R} \to \mathbb{R}$ (number of such homomorphisms is $2^{continuum}$). A composition of ψ and a flux homomorphism is non-measurable homomorphism SSymp $\to \mathbb{R}$.

We have

$$\varkappa(\mathfrak{T}_B(\varepsilon)) = \int_\Lambda F\,dx\,dy = \int_{W^{int}} + \int_{W^{ext}} + \int_{\mathfrak{U}_B(\varepsilon)}$$

The fist summand gives $\sigma(S)^2 + O(\varepsilon)$, the second summand is 0. It remains to show that F is uniformly bounded in x and ε in the strip $\mathfrak{U}(\varepsilon)$.

The value of $F(\ell(s))$ for $0 < s < 1$ is the integral of λ over a nonclosed curve L composed from $\ell(t)$ and $\mathfrak{T}(\ell(t))$ with $0 < t < s$ ($\ell(t)$ is passed in the inverse direction). We include this curve into a closed contour C adding the direct segments $[0, \ell(0)]$, $[0, \ell(s)]$. The integral of λ over these segments vanishes. Hence, by the Green formula, $F(\ell(s))$ is the oriented area of the curvilinear sector bounded by the contour C. Obviously, this area is uniformly bounded. $\square$

3.9. Preliminary remarks on trivializers. First, let us consider a domain $\Omega \subset \mathbb{R}^2$ and a map $\iota : \Omega \to \mathcal{M}$ as in 1.3. This allows to fix an explicit expression (1.5) for the cocycle C. Below we will choose Ω and ι in a certain appropriate way.

Assume, that our cocycle $C(q_1, q_2)$ is trivial on $\mathrm{Symp}(\mathcal{M}) = \mathrm{Symp}(\mathcal{M}, \Omega, \iota)$, let $\Gamma(q)$ be its trivializer. In other words, consider the space $\mathrm{Symp}(\mathcal{M}, \Omega, \iota) \times \mathbb{R}$ with the multiplication (1.6). For each $q \in \mathrm{Symp}(\mathcal{M}, \Omega, \iota)$, consider the element

$$\widetilde{q} := (q, \Gamma(q)) \in \mathrm{Symp}(\mathcal{M}, \Omega, \iota) \times \mathbb{R}$$

Then

$$\widetilde{q}_1 \widetilde{q}_2 = \widetilde{q_1 q_2}$$

Lemma 3.10 *For a diffeomorphism $q \in \mathrm{Symp}(\mathcal{M}, \Omega, \iota)$ consider the set* $\mathrm{Move}(q)$ *of all $x \in \Omega$ such that $q(x) \neq x$. Then, for each $r \in \mathrm{Symp}(\mathcal{M}, \Omega, \iota)$,*

$$|C(q, r)| < \frac{\pi}{2}\sigma(\mathrm{Move}(q)), \qquad |C(r, q)| < \frac{\pi}{2}\sigma(\mathrm{Move}(q))$$

In particular, the value of the cocycle $C(q_1, q_2)$ is $O(\varepsilon)$ if one of the arguments q_1, q_2 is an ε-twist.

Proof. Let $g_1, g_2 \in \mathrm{Sp}(2n, \mathbb{R})$. If $g_1 = 1$ or $g_2 = 1$, then $c(g_1, g_2) = 0$, see formula (2.10). It remains to apply Theorem 2.1.d.

Corollary 3.11

$$\left|\Gamma(g_1 \dots g_k) - (\Gamma(g_1) + \dots + \Gamma(g_k))\right| \leq \frac{\pi}{2} \sum \sigma_i(\mathrm{Move}(g_i))$$

Lemma 3.12 *There is a constant H such that*

$$\Gamma(\mathfrak{T}_C(\varepsilon)) = H + O(\varepsilon), \qquad \varepsilon \to 0$$

for each Dehn ε-twist.

PROOF. *Asymptotics for a single twist.* Consider a nonseparating non self-intersecting curve C on the surface. Consider a ring $\Theta \supset C$, identify this ring with a subdomain in the flat circle Λ. Our cocycle must be trivial on the group $\mathrm{Symp}(\Theta)$ of symplectomorphisms of Θ. On another side, our central extension admits a canonical trivialization Γ° on the group $\mathrm{SSymp}(\Lambda) \supset \mathrm{Symp}(\Theta)$. Hence

$$\Gamma(q) = \Gamma^\circ(q) + u(q), \qquad q \in \mathrm{Symp}(\Theta)$$

where u is a homomorphism $\mathrm{Symp}(\Theta) \to \mathbb{R}$).

The group $\mathrm{Symp}(\Theta)$ is a semidirect product $\mathbb{Z} \ltimes \mathrm{SSymp}(\Theta)$. Let $\xi \in \mathrm{Symp}(\Theta)$ be a generator of the mapping class group $\mathrm{Symp}(\Theta)/\mathrm{SSymp}(\Theta)$. Then $q = \xi^n \circ q^*$, where $q^* \in \mathrm{SSymp}(\Theta)$. Hence the homomorphism $u(q)$ must have a form

$$u(q) = n \cdot u(\xi) + a\tau(q^*) + b\varkappa(q)$$

Now let $q = \mathfrak{T}_C(\varepsilon)$. Then all the terms of $\Gamma(q)$ have asymptotics of the form $\mathrm{const} + O(\varepsilon)$ (for Γ°, see Lemma 3.6; for $\varkappa$, see Lemma 3.9; for τ this is more-or-less obvious).

Coincidence of asymtotics for different twists. Let B, C be two nonseparating non self-intersecting curves on $\mathcal{M}$. Let us show, that there exists $r \in \mathrm{Symp}(\mathcal{M})$ such that

$$\mathfrak{T}_C(\varepsilon) = r\mathfrak{T}_B(\varepsilon)r^{-1}$$

for sufficiently small ε. Indeed, by definition of ε-twists, we have fixed diffeomorphisms h_B, h_C from some ring $1-\delta < r < 1+\delta$ to some strips $\mathfrak{U}_B(\delta)$, $\mathfrak{U}_C(\delta)$ about B, C. For $m \in \mathfrak{U}_B(\delta)$, we define r as $r(m) = h_C \circ h_B^{-1}(m)$. After this we extend r to $\mathcal{M} \setminus \mathfrak{U}_B$ in an arbitrary way. This is possible since $\mathcal{M} \setminus \mathfrak{U}_B$ and $\mathcal{M} \setminus \mathfrak{U}_B$ are symplectomorphic.

Now consider the corresponding elements of $\mathrm{Symp}(\mathcal{M}, \Omega, \iota)$. We preserve for them the same notation. We have

$$\widetilde{\mathfrak{T}}_C(\varepsilon)\,\widetilde{r} = \widetilde{\mathfrak{T}_C(\varepsilon)r} = \widetilde{r\mathfrak{T}_B(\varepsilon)} = \widetilde{r}\,\widetilde{\mathfrak{T}}_B(\varepsilon)$$

Evaluating the trivializer for the first and the last terms of this chain we obtain

$$\Gamma(\mathfrak{T}_C(\varepsilon)) + \Gamma(r) + C(\mathfrak{T}_C(\varepsilon), r) = \Gamma(r) + \Gamma(\mathfrak{T}_B(\varepsilon)) + C(r, \mathfrak{T}_B(\varepsilon))$$

By Lemma 3.10, we have $\Gamma(\mathfrak{T}_C(\varepsilon)) - \Gamma(\mathfrak{T}_B(\varepsilon)) = O(\varepsilon)$. □

3.10. Proof of Theorem 1.5. Now consider an open domain $\widehat{\Delta}$ on $\mathcal{M}$ homeomorphic to a disk with 3 holes. Assume also that the set $\mathcal{M} \setminus \widehat{\Delta}$ is connected. Since the genus is ≥ 3, this is possible. Now we define more precisely the set $\Omega \subset \mathbb{R}^2$ and the map ι. Let ι identify symplectomorphically $\widehat{\Delta}$ and some disk Δ with 3 holes on $\mathbb{R}^2$; denote by Λ the simply connected domain inside the exterior boundary of Δ.

On $\mathcal{M} \setminus \widehat{\Delta}$ we can choose the map ι in an arbitrary way.

Consider 7 curves V_0, V_1, V_2, V_3, W_1, W_2, W_3 as on Fig.2.

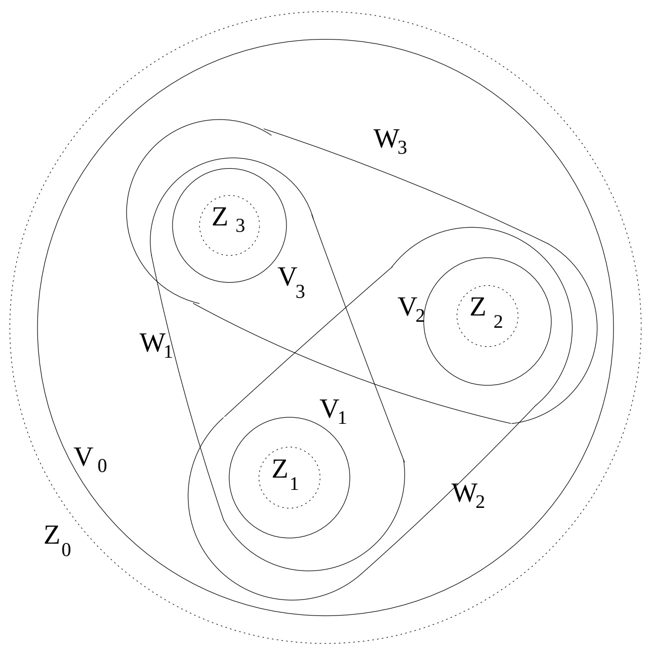

Figure 2: Notation for curves. The domain Δ is bounded 4 dotted circles Z_j. The domain Λ is the disk bounded by Z_0.

Consider the corresponding ε-twists. The diffeomorphism

$$p(\varepsilon) := \mathfrak{T}_{V_0}\mathfrak{T}_{V_1}\mathfrak{T}_{V_2}\mathfrak{T}_{V_3}\mathfrak{T}_{W_1}^{-1}\mathfrak{T}_{W_2}^{-1}\mathfrak{T}_{W_3}^{-1}$$

is isotopic to the identity map, and moreover, the isotopy can be done inside the domain Δ; this statement is the Dehn's *latern relation* in Teichmuller group rediscovered by Johnson, see [8], [19], [18]). $\square$

All our curves are nonseparating in $\mathcal{M}$, and by Lemma 3.12 the trivializer Γ on each of our 7 twists is $H + O(\varepsilon)$. By Lemma 3.10,

$$\Gamma(p) = H + O(\varepsilon), \qquad \varepsilon \to 0$$

In particular, this means that the main tern of the asymptotics of $\Gamma(p(\varepsilon))$ is invariant with respect to deformations of our seven curves $V_0, \ldots$.

On another hand, we have the trivializer Γ° of $C(\cdot,\cdot)$ on $\mathrm{SSymp}(\Delta)$ described in 2.11 and given by (3.7). The difference of two trivializers is a homomorphism $\mathrm{SSymp}(\Delta) \to \mathbb{R}$. Hence it must be a linear combination of 3 flux homomorphisms τ_j and the Calabi invariant. Thus, we have

$$-H + \Gamma^\circ(p) + \sum_{j=1}^{3} a_j \tau_j(p) + b \cdot \varkappa(p) = O(\varepsilon) \tag{3.12}$$

where a_j, b are some real constants. We evaluate $\tau_j(p)$ and $\varkappa(p)$ using Lemmae 3.8 and 3.9. Denote by $\sigma[V_i]$ (resp. $\sigma[W_j]$) the area surrounded by a contour V_i (resp. W_j). Then

$$\begin{aligned}
&-H + 2\pi \sum_i \sigma[V_i] - 2\pi \sum_j \sigma[W_j] + \\
&+ a_1(\sigma[V_0] + \sigma[V_1] - \sigma[W_1] - \sigma[W_2]) + a_2(\sigma[V_0] + \sigma[V_2] - \sigma[W_2] - \sigma[W_3]) + \\
&\qquad + a_3(\sigma[V_0] + \sigma[V_3] - \sigma[W_3] - \sigma[W_1]) + \\
&\qquad\qquad + b\Big(\sum_i \sigma[V_i]^2 - \sum_j \sigma[W_j]^2\Big) = O(\varepsilon)
\end{aligned}$$

We can vary the areas $\sigma[\cdot]$ in arbitrary way in certain small intervals. This is a contradiction.

4 Nontriviality of the extension in the case of tori

4.1. Formulation of result. Consider the torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. In notation of 1.3, it is natural to consider $\Omega = (0,1)^{2n}$ and the identical embedding $\Omega \to \mathbb{R}^n \to \mathbb{T}^{2n}$. Then Theorem 1.6 is a corollary of the following theorem.

Theorem 4.1 *For $n > 2$, the $\mathbb{R}$-valued cocycle $c \in H^2(\mathrm{Sp}(2n,\mathbb{R}))$ is nontrivial on the group* $\mathrm{Sp}(2n,\mathbb{Z})$.

Fix $\alpha > 0$. Consider the homomorphism $\mathbb{R} \to \mathbb{R}/2\pi\alpha\mathbb{Z}$ and consider the image c_α of c under this map.

Theorem 4.2 *For $n > 2$ and a noninteger $\alpha > 0$, the $\mathbb{R}/2\pi\alpha\mathbb{Z}$-valued cocycle c_α is nontrivial on the group* $\mathrm{Sp}(2n, \mathbb{Z})$.

Theorem 4.2 implies Theorem 4.1, thus it is sufficient to prove Theorem 4.2.

REMARK. For integer α the cocycle c_α is trivial on the whole group $\mathrm{Sp}(2n, \mathbb{R})$.

REMARK. Consider the subgroup $\Gamma_{1,2} \subset \mathrm{Sp}(2n, \mathbb{Z})$ consisting of matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that diagonals entries of the matrices A^tC, B^tD are even. The cocycle $c_{1/2}$ is trivial on $\Gamma_{1,2}$ (see [23], Section II.5),

4.2. Realization of the cocycle c_α in a representation. Denote by B_n the set of complex symmetric ($z = z^t$) matrices with norm < 1 (the Cartan matrix ball).

The symplectic group acts on B_n by the linear-fractional transformations

$$z \mapsto z^{[g]} = (\Phi + z\overline{\Psi})^{-1}(\Psi + z\overline{\Phi})$$

Fix $\alpha > 0$. Consider the representation $\widetilde{T}_\alpha$ of $\mathrm{Sp}(2n, \mathbb{R})$ in the space of holomorphic functions on B_n given by

$$\widetilde{T}_\alpha(g) f(z) = f(z^{[g]}) \det(\Phi + z\overline{\Psi})^{-\alpha} \tag{4.1}$$

It is a standard formula for a highest weight representation of the group $\mathrm{Sp}(2n, \mathbb{R})$.

For $z \in \mathrm{B}_n$, the matrix $\Phi + z\overline{\Psi}$ is nondegenerat, see (2.14). Hence the expression

$$\det(\Phi + z\overline{\Psi})^{-\alpha} = \det \Phi^{-\alpha} \det(1 + z\overline{\Psi}\Phi^{-1})^{-\alpha}$$

has countable number of branches; they are enumerated by values of $(\det \Phi)^{-\alpha}$. Thus $\widetilde{T}_\alpha$ is a projective representation of $\mathrm{Sp}(2n, \mathbb{R})$ (or representation of the universal covering group $\mathrm{Sp}(2n, \mathbb{R})^\sim$).

Nevertheless, we interpret the standard formula (4.1) in the following slightly nonstandard way. Let us consider the normalized operators $T_\alpha(g)$ given by

$$T_\alpha(g) = f(z^{[g]}) \det(1 + z\overline{\Psi}\Phi^{-1})^{-\alpha}$$

The expression

$$(1 + z\overline{\Psi}\Phi^{-1})^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{k!} \big(-\overline{\Psi}\Phi^{-1}\big)^k$$

is well defined as a sum of series. Thus its determinant is well defined.

Proposition 4.3 .

$$T_\alpha(g_1) T_\alpha(g_2) = \sigma_\alpha(g_1, g_2) T_\alpha(g_1 g_2) \tag{4.2}$$

where

$$\sigma_\alpha(g_1, g_2) = \det\Big[\Phi(g_1)^{-1}\Phi(g_1 g_2)\Phi(g_2)^{-1}\Big]^{-\alpha} = \exp\{-\alpha c(g_1, g_2)\} \tag{4.3}$$

Proof. For g_1, g_2 near 1, it is proved by a trivial calculation. After this we consider the analytic continuation in g_1, g_2. $\square$

The cocycles σ_α are precisely the cocycles c_α in multiplicative notation. We intend to prove Theorem 4.2 in the following form:

the restriction of the representation T_α to $\mathrm{Sp}(2n,\mathbb{Z})$ can not be reduced to a linear representation of $\mathrm{Sp}(2n,\mathbb{Z})$ by a correction of the form

$$T_\alpha(g) \mapsto \gamma(g)T_\alpha, \qquad \gamma(g) \in \mathbb{C}^*$$

where g ranges in $\mathrm{Sp}(2n,\mathbb{Z})$.

4.3. Another model of the same representation. By W_n we denote the Siegel wedge, i.e., the set of complex symmetric matrices satisfying z with $\mathrm{Im}\, z > 0$.

The group $\mathrm{Sp}(2n,\mathbb{R})$ acts on W_n by the transformations

$$z \mapsto z^{[g]} := (a + zc)^{-1}(b + zd)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a real symplectic matrix in the usual notation. The action of $\mathrm{Sp}(2n,\mathbb{R})$ on W_n is given by

$$S_\alpha(g)f(z) = f(z^{[g]}) \det(a + zc)^{-\alpha} \tag{4.4}$$

It is well-known, that the (projective) representation S_α is equivalent to representation T_α defined above. The intertwining operator is given by the transformation

$$T_\alpha\Big[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\Big] f(z) := f((1 + iz)^{-1}(i + z)) \cdot \det(1 + iz)^{-\alpha} \cdot 2^{\alpha/2}$$

where $z \in B_n$ (then its Cayley transform $(1 + iz)^{-1}(i + z)$ is an element of W_n).

4.4. Linearization of representation on a upper triangular subgroup. Consider the subgroup $\mathcal{B}(\mathbb{Z}) \subset \mathrm{Sp}(2n,\mathbb{Z})$ consisting of the matrices

$$\begin{pmatrix} A & B \\ 0 & A^{t-1} \end{pmatrix}; \qquad \det A = 1$$

The cocycle σ_α is equivalent to trivial cocycle on this subgroup. Indeed, the operators

$$S_\alpha \begin{pmatrix} A & B \\ 0 & A^{t-1} \end{pmatrix} f(z) = f\big(A^{-1}(B + zA^{t-1})\big) \tag{4.5}$$

define a linear representation of $\mathcal{B}(\mathbb{Z})$.

Lemma 4.4 *Formula (4.5) gives a unique possible linearization of the representation S_α on the subgroup $\mathcal{B}(\mathbb{Z})$.*

This follows from the next lemma.

Lemma 4.5 *The group $\mathcal{B}(\mathbb{Z})$ has no homomorphisms to $\mathbb{C}^*$. In particular, it has no homomorphisms to $\mathbb{Z}$ and $\mathbb{Z}_k$.*

PROOF. Let $\chi : \mathcal{B}(\mathbb{Z}) \to \mathbb{C}^*$ be a homomorphism.

First, the group $\mathcal{B}(\mathbb{Z})$ contains the group $\mathrm{SL}(n, \mathbb{Z})$ consisting of matrices $\begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}$. This group has no Abelian quotients. Hence, $\chi = 1$ on $\mathrm{SL}(n, \mathbb{Z})$.

Also, $\mathcal{B}(\mathbb{Z})$ contains the group $N(\mathbb{Z})$ consisting of matrices $\nu(B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$; the product in this group corresponds to the sum of the matrices B:

$$\begin{pmatrix} 1 & B_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & B_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & B_1 + B_2 \\ 0 & 1 \end{pmatrix}$$

Also

$$\begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & ABA^t \\ 0 & 1 \end{pmatrix}$$

Thus, we must prove, that the there is no characters

$$\chi(B_1 + B_2) = \chi(B_1)\chi(B_2)$$

on the additive group of symmetric integer matrices B such that

$$\chi(ABA^t) = \chi(B) \qquad \text{for } A \in \mathrm{SL}(n, \mathbb{Z})$$

Any character of $N(\mathbb{Z})$ has the form

$$\chi(B) = \prod_{i \geq j} y_{ij}^{b_{ij}}, \qquad y_{ij} \in \mathbb{C}^*$$

Since the group $\mathrm{SL}_n(\mathbb{Z})$ contains all the even permutations of coordinates, our character has the form

$$\chi(B) = u^{\mathrm{tr}\ B} \cdot v^{\sum_{i>j} b_{ij}}$$

Next,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} a_{11} + 2a_{12} + a_{22} & a_{12} + a_{22} & a_{13} + a_{23} \\ a_{12} + a_{22} & a_{22} & a_{23} \\ a_{13} + a_{23} & a_{23} & a_{33} \end{pmatrix}$$

Hence, for all $a_{ij} \in \mathbb{Z}$ we have

$$u^{2a_{12} + a_{22}} v^{a_{22} + a_{23}} = 1$$

Thus $u = v = 1$. $\square$

4.5. Proof of Theorem 4.2. Assume that we have some linearization S_α° of the representation S_α on $\mathrm{Sp}(2n, \mathbb{R})$. By Lemma 4.4, this linearization is rigidly defined on the matrices $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$ by the formula

$$S_\alpha^\circ \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(z) = f(z + B) \tag{4.6}$$

Now let us consider the subgroup $\mathrm{SL}(2, \mathbb{Z}) = \mathrm{Sp}(2, \mathbb{Z}) \subset \mathrm{Sp}(2n, \mathbb{Z})$ consisting of the $[1 + (n-1) + 1 + (n-1)] \times [1 + (n-1) + 1 + (n-1)]$ matrices

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To be short, below we will write $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let us show that having condition (4.6), we can not trivialize the cocycle on $\mathrm{SL}(2, \mathbb{Z})$.

Denote

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

It can be easily be checked that

$$I^4 = 1, \quad J^3 = 1, \quad IK = J \tag{4.7}$$

Our operators $S_\alpha^\circ(\cdot)$ have the form

$$\begin{aligned} S_\alpha^\circ(K) f(z) &= f(z - 1), \\ S_\alpha^\circ(I) f(z) &= \theta \cdot f(-1/z) z^{-\alpha}, \\ S_\alpha^\circ(J) f(z) &= \theta' \cdot f(-1 - 1/z) z^{-\alpha} \end{aligned}$$

where

$$z^{-\alpha} = |z|^{-\alpha} \exp(-i\alpha \arg z); \qquad 0 < \arg z < \pi$$

and θ, $\theta' \in \mathbb{C}^*$ are some unknown constants. The equation $IK = J$ implies $\theta' = \theta$. Also

$$S_\alpha^\circ(I)^4 = \theta^4 \exp(-2\alpha\pi i), \qquad S_\alpha^\circ(J)^3 = \theta^3 \exp(-2\alpha\pi i) \tag{4.8}$$

Since the both these operators equal 1, $\theta = 1$, $\exp(-2\alpha\pi i) = 1$. We obtaine a contradiction, since $\alpha \notin \mathbb{Z}$.

Remark. An evaluation of powers in (4.8) can be simplified in the following way. First, the point i is a fix point of the transformation $z \mapsto -1/z$. Hence we can follow only the values of $S_\alpha^\circ(I)^k f(z)$ in this point. In the second case the fixed point is $\lambda = \exp(2\pi i/3)$.

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